An easy-to-hedge covariance swap

Covariance swaps that track the covariance of assets can be difficult to hedge because there is no known static replication formula, unlike the case for variance swaps. However, in the case of swaps measuring the covariance between the absolute returns of an equity and a foreign exchange rate, it is possible to hedge the contract using a static hedge in two equity quanto forwards and a dynamic hedge in the stock and the forex rate. By Per Hörfelt, Sudhansu Kajbaje, Davit Sahakyan and Jinguo Zhao

Since the seminal papers by Neuberger (1990) and Dupire (1993), contracts on volatility and correlation have been of increasing interest. Volatility and correlation products are actively traded on exchanges and over-the-counter. In particular, contracts that pay the realised variance of the log returns of the underlying asset have become a popular product. Part of the reason for their popularity is that they can be perfectly hedged with a static portfolio of European-style call and put options, independent of the volatility of variance. For further details, see Carr & Madan (1999).

Covariance swaps are a generalisation of the variance swap. They pay the realised covariance between the log returns of two underlying assets. They have not reached the same popularity as variance swaps, because there is no known static replication formula, and the hedge depends on the volatility of variance and the correlation of the underlying.

Furthermore, hedging these quantities may be expensive. For instance, consider a daily observed covariance swap between the S&P 500 index \( X(t) \) and the euro/dollar exchange rate \( X(t) \), that is, \( X(t) \) is the value of one euro in US dollars. This pays:

\[
\sum_{n=1}^{252} \ln \left( \frac{s(t_i)}{s(t_{i+1})} \right) \ln \left( \frac{X(t_i)}{X(t_{i+1})} \right)
\]

It is natural to hedge the cross gamma risks with quanto forwards that have the payoff function \( s(t)X(t) \) in dollars at time \( t \). There is a broker market for these contracts. Suppose \( \gamma_s(x,s) \) and \( \gamma_Q(x,q) \) are the price of the covariance swap and the quanto forward, respectively, as a function of the stock price \( s \) and the exchange rate \( q \). Assuming time to maturity is one year, that is, \( n = 252 \), it is reasonable to approximate:

\[
\frac{\partial^2 \gamma_s}{\partial x^2} = \frac{1}{sx} \quad \text{and} \quad \frac{\partial^2 \gamma_Q}{\partial q^2} = 1
\]

Thus, to hedge the cross-gamma term, the seller of the covariance swap needs to keep \( 1/sx \) in quanto forwards. If the stock increases by \( d \), the amount in quantos will have to decrease by roughly the same proportion. Suppose the stock has daily moves of 2% and the bid-ask spread for the quanto is 20 basis points. Then the cost incurred due to the bid-ask spread is around 0.20bp each day. Over a year that amounts to around 50bp, which is a significant cost. There are other potential issues with trading covariance swaps. It is not clear how to hedge the volatility of variance for the equity or the foreign exchange rate. For a further discussion on the hedging strategy of covariance swaps, see Jamshidian (1994).

This article discusses a slight modification of the traditional covariance swaps that we will refer to as Emilie (covariance with Minimal replication Expense’. Instead of measuring the covariance between the log-returns, it pays the covariance between the absolute returns. If the underlyings are a stock and a forex rate and the quanto forward is tradable, it turns out that this modified covariance swap does not suffer from any of the drawbacks of the traditional covariance swap. Indeed, it can be almost perfectly hedged by keeping a static number of quanto forwards and a dynamically rebalanced portfolio in equity and forex futures. The hedging portfolio does not require taking any positions in options on the equity or forex rate.

Among the potential buyers of this modified covariance swap are hedge funds. During summer 2010, the price differential between the euro value of S&P forwards and the quanto equivalent was historically high. This attracted hedge funds to bet on this spread tightening. However, to extract this spread, the portfolio of S&P and currency futures and quantos needed to be dynamically hedged for moves in currency and spot, something many hedge funds wanted to avoid. Investing in a covariance swap gives a cleaner exposure to the price differential without the extra delta risk and would therefore be preferred by many hedge funds.

The pricing and replication of products linked to realised covariance has been discussed previously in literature. Bossu (2005) and Brockhaus & Giese (2010) discuss the pricing of correlation swaps under different model assumptions. The replication of covariance swaps is discussed in Bossu (2006), Carr & Corso (2001), Carr & Madan (1999) and Jamshidian (1994). The product that we will present here is similar to the covariance contract in Carr & Corso (2001), which also derives a static replication formula. However, the replication is based on a strip of options on the spread between the underlyings. To the best of our knowledge, there exists no broker market for options on the spread between a stock and an exchange rate, not even for at-the-money options. Hence, the replication strategy in Carr & Corso (2001) is more of academic interest. For some further statistical results on covariance swaps, see Fonseca, Ielp & Grasselli (2008).

This article starts by discussing the payout of the contract. We then discuss how the contract can be hedged if the underlying price processes are futures. Finally, we consider the replication strategy if the price processes are spot prices and give some numerical examples displaying the accuracy of the hedge.

Emilie

Consider a contract that is initiated at \( T_0 \) and has \( n \) observation dates \( t_i \) between \( T_0 \) and \( T_2 \), which pays:

\[
\chi = \sum_{i=0}^{n-1} [Y(t_i) - Y(t)] [Z(t_i) - Z(t)]
\]

where \( Y \) and \( Z \) are price processes for two different assets. We will refer to this contract as Emilie. This article will focus on the special case when one of the assets is a stock or stock future and the other is an exchange rate or exchange rate future.

It is natural to normalise the contract with the factor:

\[
\frac{N}{n} Y(T_0) Z(T_0)
\]
where $N$ is the number of monitoring times per year. Typically there will be daily monitoring and $N = 252$. Covariance swaps usually measure the covariance between log returns, whereas Emilie pays the covariance between the absolute returns of the underlying assets.

**Emilie on futures**

Let $S(t)$ be the level of the S&P at time $t$ and $X(t)$ be the euro/dollar exchange rate, that is, $X(t)$ is the value of one euro in US dollars. In addition, let $F(t, T)$ and $F_q(t, T)$ be the level at time $t$ of futures on the S&P and the euro/dollar rate, respectively, maturing at $T$. Suppose $r(t)$ is the deterministic dollar short rate observed at $t$. Finally, assume it is possible to trade quanto forwards maturing at $T_i$ and $T_j$, that is, contracts that pay $S(T_i)X(T)$ in dollars at time $T$ for $i = 1, 2$.

Let $y_i$ and $z_i$ be a sequence of real numbers and note:

\[
(y_{i+1} - y_i)(z_{i+1} - z_i) = z_{i+1}y_{i+1} - z_iy_{i} - z_{i+1}y_{i} + y_i(z_{i+1} - z_i)
\]

so that:

\[
\sum_{i=0}^{n-1}(y_{i+1} - y_i)(z_{i+1} - z_i) = z_ny_n - z_0y_0 - \sum_{i=0}^{n-1}z_i(y_{i+1} - y_i) - \sum_{i=0}^{n-1}y_i(z_{i+1} - z_i)
\]

Setting $y_i = F(t_i, T)$ and $z_i = F_q(t_i, T_q)$, we get that an Emilie on futures can be hedged by:

- going short $\exp(-\int_{T_i}^{t} r(\tau)d\tau)F(\tau, T)$ for each time $t_i$ and exiting each trade at the next monitoring time $t_{i+1}$. The proceeds are invested in cash.
- going short $\exp(-\int_{T_i}^{t} r(\tau)d\tau)F_q(\tau, T_q)$ in forex futures at each time $t_i$ and exiting at the next monitoring time $t_{i+1}$. The proceeds are invested in cash.
- going long, at time $T_i$, one quanto expiring at $T_j$ and short $\exp(-\int_{T_i}^{t} r(\tau)d\tau)$ in quantos maturing at $T_i$. The quanto maturing at $T_i$ is kept to its expiry, whereas the payout from the quantos maturing at $T_i$ are invested in cash between $T_i$ and $T_j$.

So far we have assumed that the interest rate is deterministic. If we were to relax this condition, we can note that the number of futures in the replication portfolio at any time $t_i$ depends on the discount factor between $t_{i-1}$ and $T_i$, which is not known at time $t_i$. Thus, there is an interest rate risk but it will typically be small if the contract has daily observation dates.

**Emilie on spot**

If the underlyings are spot prices instead of future prices, we cannot apply the argument in the previous section due to the funding cost that appears when trading spot. However, this section will show that the payout from an Emilie is approximately the same whether the underlying price processes are spot or future prices, at least up to a factor that only depends on the interest rate and dividends. Thus, an Emilie on spot can be hedged following the scheme outlined in the previous section.
This section is divided into two parts. The first will motivate the claim that the payout from an Emili on spot is approximately equal to the payout from a certain number (denoted $\alpha$) of Emilies on futures, at least if we restrict the underlying price processes to be Itô diffusions, assume that the instantaneous correlations do not change sign, and consider a continuously monitored Emili, that is, keep $T_i$ and $T_j$ fixed and let $n \to \infty$ in equation (3). The second part of this section will give some numerical examples giving further evidence that the error in this estimate between Emili on spot and futures is small, even if we consider a daily monitored Emili and allow the instantaneous correlation to change sign.

Consider the special case when the underlying price processes are Itô diffusions and let $n \to \infty$ in the payout in equation (3). To be more specific, suppose:

$$\frac{dS(t)}{S(t)} = \left(r_e(t) - d(t)\right) dt + \sigma_s(t) dW_i(t)$$

and:

$$\frac{dX(t)}{X(t)} = \left(r_e(t) - r_i(t)\right) dt + \sigma_s(t) \left[\rho(t) dW_i(t) + \sqrt{1 - \rho^2(t)} dW_2(t)\right]$$

where $r_e$ is the deterministic euro rate, $d$ is the deterministic dividend yield for S&P, $\rho$ is a stochastic process bounded by minus one and one, $W_i(t), W_j(t)$ are standard independent Brownian motions, and $\sigma_s, \sigma_r$ are stochastic processes. Note that if the underlying stock pays fixed dividends, the effect of these can be stripped out. Their contribution to the payout will be an additional term of the form $\sum \delta_i X(t_{i-1}) - X(t_i)$, where $\delta_i$ is the fixed dividend at time $t_i$. It is evident how to hedge this sum, and it is therefore no loss of generality to exclude fixed dividends from the discussion. Moreover, let $n \to \infty$ in equation (3) so that the payout of an Emili on spot, from now on denoted $\chi$, equals:

$$\chi = \int_{t_i}^{T_1} S(t) X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt$$

(9)

whereas for the same contract on futures $\chi_f$ is:

$$\chi_f = \int_{t_i}^{T_1} F_S(t) F_X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt$$

(10)

Hence, the difference between the payouts $\chi$ and $\chi_f$ weighted by a factor $\alpha$ equals:

$$\chi - \alpha \chi_f = \int_{t_i}^{T_1} \left[1 - \alpha \rho(t) \int \chi(t) X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt\right] S(t) X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt$$

(11)

Thus, if the correlation process $\rho$ does not change sign, the relative difference is bounded by:

$$\left|\frac{\chi - \alpha \chi_f}{\chi_f}\right| \leq \max_{t_i \leq t \leq T_1} \left|1 - \alpha \rho(t) \int \chi(t) X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt\right|$$

(12)

To minimise the difference between $\chi$ and $\alpha \chi_f$, it is natural to put $\alpha = \alpha^*$, where:

$$\alpha^* = e^{\int \chi(t) X(t) \sigma_s(t) \alpha(t) \rho(t) \, dt}$$

(13)

and:

$$\tau = \frac{T_1 + T_2}{2}$$

(14)

With this choice of $\alpha$, the right-hand side in equation (12) will be small, in particular with the present level of interest rates. Typically, the error relative to the price is around a few per cent. For instance, in the numerical example in figure 1, the relative error is bounded by 6%. The price is around 2%, so the error relative to the notional amount is bounded by 12bp. This number can be compared with the bid-ask spread for a quanto option, which in a normal market is around 10bp.
The back testing is performed for 1,000 different paths. The figures December 2008). The table shows that the payouts can be significantly different December. The payouts are calculated from real market data.

Similar back testing on real market data typically shows a smaller error, since the interest rate differential is smaller than what has been used in the numerical examples in this document.

<table>
<thead>
<tr>
<th>3 Hedging error as a percentage of the initial price (weighted hedge, factor $\alpha = \alpha'$, euro rate $r_e = 1%$, dollar rate $r_d = 5%$ and dividend yield $d = 0%$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^2_{t}(t) = \kappa_{t} \left( u_{\sigma} - \sigma(t)^2 \right) dt + \nu_{\sigma} \sigma(t) dW_{\sigma}(t)$</td>
</tr>
<tr>
<td>$d \rho(t) = \kappa_{\rho} \left( u_{\rho} - \rho(t) \right) dt + \nu_{\rho} \sqrt{1 - \rho^2(t)} dW_{\rho}(t)$</td>
</tr>
</tbody>
</table>

The stochastic differential equation for $\rho$ is sometimes referred to as a Jacobi process and it has been discussed previously (see, for example, Brockhaus & Giese, 2010). The values of the parameters are shown in Table A.

Figures 1, 2, 3 and 4 display histograms describing the hedging errors for different levels of interest rates and dividends and for different choices of $\alpha$, that is, the contract has been hedged as if the payout equals $\alpha$ number of Emilies on futures. Figures 1 and 3 have $\alpha = \alpha'$ and figures 2 and 4 have $\alpha = 1$. All examples consider a daily monitored Emiae written on spot with one year to maturity, that is, $n = N = 252$, $T_{1} = T_{2}$ and $T_{2} - T_{1} = 1$ year.

The error is presented relative to the initial price for the contract. The back testing is performed for 1,000 different paths. The figures show that the hedging error is small, at least if we hedge an Emiae on spot as if it is written on futures times the optimal weight $\alpha'$. Similar back testing on real market data typically shows a smaller error, since the interest rate differential is smaller than what has been used in the numerical examples in this document.

The last numerical example (see Table B) illustrates the difference in final payout between a standard covariance swap and Emiae. The example compares the payouts over four different time periods, all stretching over one year and maturing in December. The payouts are calculated from real market data. The table shows that the payouts can be significantly different (see, for instance, the time period from December 2007–December 2008).

Moreover, if the trade is long-dated and stretching over several years, then the payout can be split into tranches, with each tranche stretching from one liquid maturity in the quanto market to the next. So, by dividing the payout into different tranches the error in the hedging can be reduced.

<table>
<thead>
<tr>
<th>4 Hedging error as a percentage of the initial price (non-weighted hedge, factor $\alpha = 1$, euro rate $r_e = 5%$, dollar rate $r_d = 0.5%$ and dividend yield $d = 0%$)</th>
</tr>
</thead>
</table>

Conclusion
In this article, we have discussed a modification of the traditional covariance swap. The contract pays the covariance between the absolute returns of two underlying assets. If one underlying is a stock future and the other is a forex future, we show that it is possible to perfectly hedge the contract with a static position in the quanto forward and a dynamic portfolio in stock and forex futures. Moreover, the dynamic portfolio is independent of the volatility and correlation of the stock and forex rate. Furthermore, if the underlyings are spot prices on a stock and a forex rate, we describe a hedging strategy that is also static in the quanto forward but is not exact. The mismatch in the hedging strategy is small, however, something that is motivated both theoretically by considering a continuous version of the covariance swap and practically by presenting some numerical examples.

Per Hörfelt is a quantitative analyst at Royal Bank of Canada in London, Sudhansu Kajbaje is a structured trader and Davit Sahakyan is a quantitative analyst at Barclays in New York, and Jinguo Zhao is a PhD student at Courant Institute of Mathematical Sciences in New York. The content of this article represents the authors’ personal opinions and does not reflect the views of Barclays or Royal Bank of Canada. The authors would like to thank Olaf Torné for his comments and suggestions. Email: pereniar.horfelt@rbccm.com, sudhansu.kajbaje@barclayscapital.com, davit.sahakyan@barclayscapital.com, jgzhao@cims.nyu.edu

References
Carr P and D Madan, 2005
A new approach for modelling and pricing correlation swaps
Global Derivatives Trading & Risk Management, May

Carr P and P Corso, 2001
Covariance contracting for commodities
Energy Risk, April, pages 42–45

Carr P and D Madan, 1999
Currency covariance contracting
Risk February, pages 47–51

Carr P and D Madan, 1999
Towards a theory of volatility trading
Handbooks in Mathematical Finance: Option Pricing, Interest Rates and Risk Management, pages 458–476

Neuberger A, 1990
Volatility trading
Working paper, London Business School

Dupire B, 1993
Model art
Risk September, pages 118–124

Fonseca J, F Ielp and M Grasselli, 2008
Hedging (co)variance risk with variance swaps
Working paper, available at SSRN

Jacquier A and S Slaoui, 2007
Variance dispersion and correlation swaps
Working paper, available at SSRN

Jamshidian F, 1994
Hedging quantos, differential swaps and ratios
Applied Mathematical Finance 1(1), pages 1–20

Barclays in New York, and Jinguo Zhao is a PhD student at Courant Institute of Mathematical Sciences in New York. The content of this article represents the authors’ personal opinions and does not reflect the views of Barclays or Royal Bank of Canada. The authors would like to thank Olaf Torné for his comments and suggestions. Email: pereniar.horfelt@rbccm.com, sudhansu.kajbaje@barclayscapital.com, davit.sahakyan@barclayscapital.com, jgzhao@cims.nyu.edu.